$D^1_+ := \{ f : C^1 \text{ diffeo of } I, \text{ increasing} \}$

$$D^1_+ \ni f \longmapsto f'(0) \in \mathbb{R}_{>0}$$

cts, conj invariant

If D^1_+ had dense conj class, then constant

$$\mathrm{id}_I \mapsto 1$$

$$\frac{x^2+x}{2} \mapsto \cancel{1}\ \frac{1}{2}$$

$C \subseteq G$ conj class

C non-meagre $\Rightarrow$ C is G_δ in G

Effros, 1965

Proven independently by Marker, 1988
and Sami 1994

Recall: $I := [a,b] \subseteq \mathbb{R}$

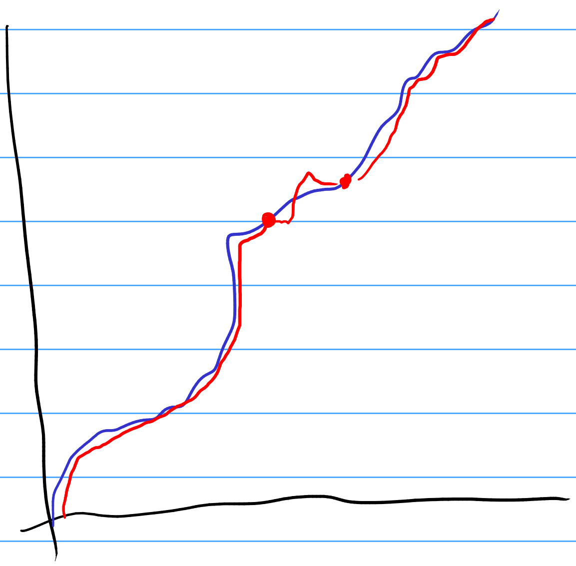
- $H_+ := \{ f \in \mathrm{Homeo}(I) : f(0) = 0 \}$

$H_+(I)$ unif cvgce top.

- $H_+^{AC}(I) := \{ f \in H_+(I) : f \text{ and } f^{-1} \text{ AC} \}$

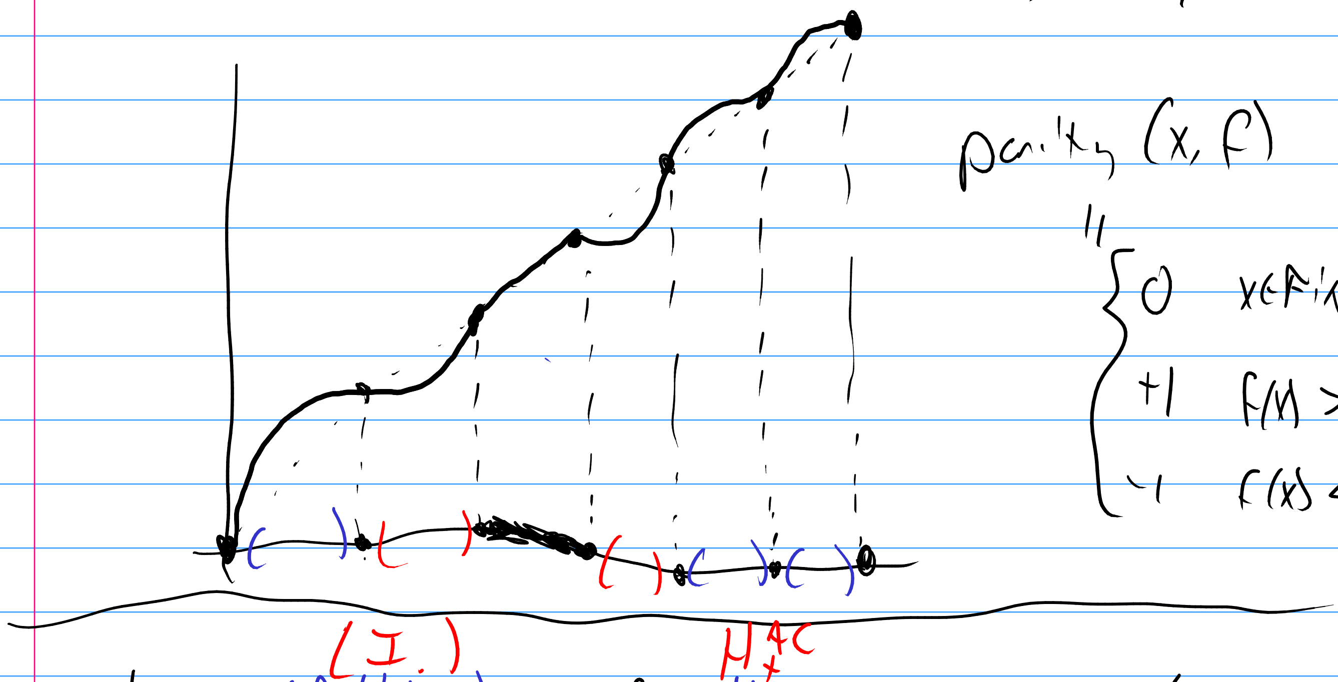
Top: $\rho_{AC}(f,g) := \int_I |f' - g'| \, dx$

$$\leq \int |f'| + \int |g'|$$
$$= \int f' + \int g'$$
$$= f(b) - f(a) + g(b) - g(a)$$
$$= 2(b-a)$$



Notation: $-$ $\mathrm{Fix}(f) := \{x \in I : f(x) = x\}$

$-$ $\mathcal{O}_f(x) := \{y \in I : \exists i,j \in \mathbb{Z}\ \ f^i(x) \leq y \leq f^j(x)\}$



$$\mathrm{parity}(x,f) = \begin{cases} 0 & x \in \mathrm{Fix}(f) \\ +1 & f(x) > x \\ -1 & f(x) < x \end{cases}$$

Thm (folklore) $f \in H_+$ is generic (i.e. has comeagre conj class) iff:

(i) $\mathrm{Fix}(f)$ is perfect and totally disconnected
($\Leftrightarrow$ $\mathrm{Fix}(f)$ homeo. to Cantor space)

$\Leftrightarrow$ $\{\mathcal{O}_f(x) : x \notin \mathrm{Fix}(f)\}$ is a dense linear order w/o endpts.

(ii) $\forall \sigma \in \{-1, +1\}$, $\{\mathcal{O}_f(x) : \mathrm{parity}(x,f) = \sigma\}$

is dense in $\{\mathcal{O}_f(x) : x \notin \mathrm{Fix}(f)\}$

(iii) $\lambda(\mathrm{Fix}(f)) = 0$

• Note $\mathrm{Fix}(hfh^{-1}) = h[\mathrm{Fix}(f)]$

So (iii) is conj invt in H_+^{AC}.

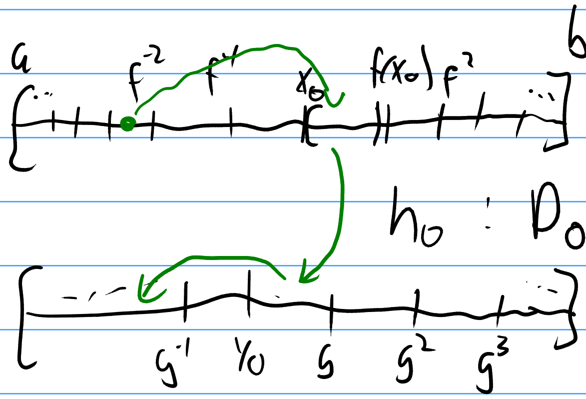
• Lemma: Let $f \in H_+^{AC}([a,b])$ and $g \in H_+^{AC}([c,d])$ such that f, g fix only the endpoints, and their nonzero orbital has the same parity. Then $\exists\, h : [a,b] \to [c,d]$ homeo s.t. $h \circ f = g \circ h$, and h, h^{-1} are AC.

Pf sketch: WLOG, f, g positive bumps.

Fix $x_0 \in (a,b)$ $\quad y_0 \in (c,d)$

Let $D_0 := [x_0, f(x_0))$, $\quad E_0 := [y_0, g(y_0))$

Note $(a,b) = \bigsqcup_{n \in \mathbb{Z}} f^n[D_0]$



$h_0 : D_0 \to E_0$ homeo st h_0 and h_0^{-1} AC

QED

Claim: If $f, g \in H_+^{AC}$ satisfy (i) – (iii), then f, g conjugate in H_+^{AC}.

Pf: Enumerate the non-zero orbitals $((a_n, b_n))_{n \in \omega}$. (so $a_n, b_n \in \mathrm{Fix}(f)$)

By (i), (ii), f, g conj in H_+.

So take $\hat{h} \in H_+$ st $\hat{h} \circ f = g \circ \hat{h}$

Observe $\hat{h}[\mathrm{Fix}(f)] = \mathrm{Fix}(g)$.

Moreover, $\hat{h}$ preserves parity of orbitals.
i.e.
$f\restriction_{[a_n, b_n]}$ and $g\restriction_{[\hat{h}(a_n), \hat{h}(b_n)]}$ satisfy

the hyp of prev lemma.

So take $h_n : [a_n, b_n] \to [\hat{h}(a_n), \hat{h}(b_n)]$ st $h_n \circ f = g \circ h_n$ $\forall n$, and h_n, h_n^{-1} AC.

$$h(x) := \begin{cases} h_n(x) & \text{if } x \in (a_n, b_n) \\ \hat{h}(x) & \text{if } x \in \mathrm{Fix}(f) \end{cases}$$

Note $h \circ f = g \circ h$.

Remains to show $h \in H_+^{AC}$.

i.e. $\forall x$ $h(x) = \int_0^x h' \, d\lambda$

(For definiteness, $I = [0,1]$)

Fix $x \in \mathrm{Fix}(f)$. Let $E := \{ n < \omega : (a_n, b_n) \subset [0,x] \}$

Note $[0,x] \setminus \mathrm{Fix}(f) = \bigsqcup_{n \in E} (a_n, b_n)$.

$$[0, h(x)] \setminus \mathrm{Fix}(g) = \bigsqcup_{n \in E} (h(a_n), h(b_n))$$

$$\int_0^x h' \, d\lambda = \int_{\bigsqcup_{n\in E} (a_n, b_n)} h' \, d\lambda + \overset{0}{\cancel{\int_{\mathrm{Fix}(f) \cap [0,x]} h' \, d\lambda}} \quad \text{by (iii)}$$

$$= \sum_{n \in E} \int_{a_n}^{b_n} h_n' \, d\lambda$$

$$= \sum_{n \in E} h_n(b_n) - h_n(a_n)$$

$$= \sum_{n \in E} \lambda\big((h(a_n), h(b_n)) \big)$$

$$= \lambda\Big(\bigsqcup_{n \in E} (h(a_n), h(b_n)) \Big)$$

$$= \lambda\big([0, h(x)] \setminus \underbrace{\mathrm{Fix}(g)}_{\text{null by (iii)}} \big)$$

$$= \lambda([0, h(x)]) = h(x).$$

Otherwise, if $x \in (a_n, b_n)$,

$$\int_0^x h' \, d\lambda = \int_0^{a_n} h' \, d\lambda + \int_{a_n}^x h' \, d\lambda$$

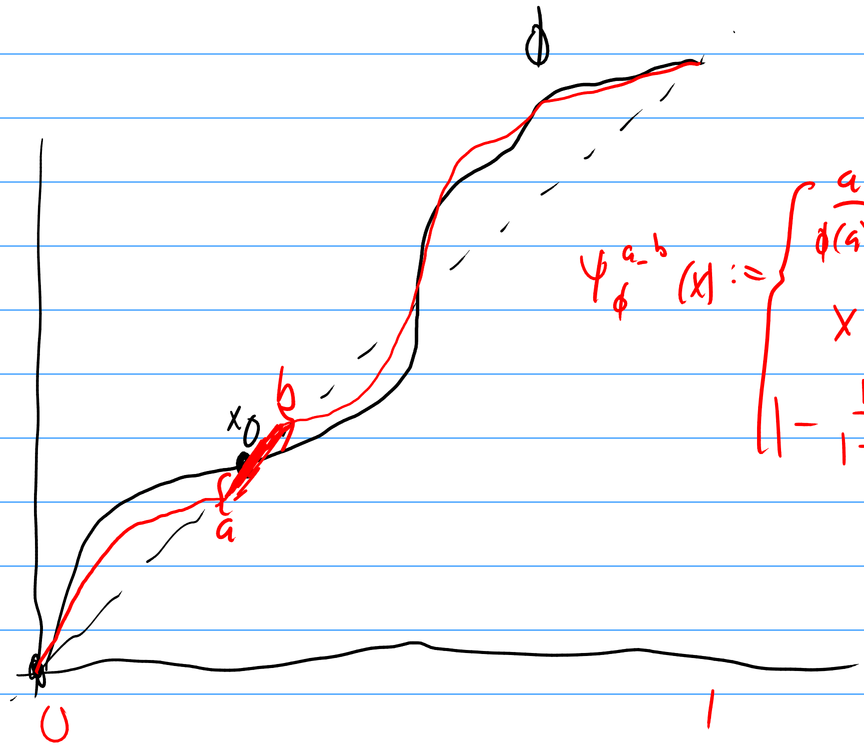
$$= h(\cancel{a_n}) + (h_n(x) - h_n(\cancel{a_n})) = h(x).$$

Same for h^{-1}, so $h \in H_+^{AC}$ QED

Remains to show (i), (ii), (iii) are cnvgr properties.

Facts:

- $\{ f \in H_+ : \lambda(\mathrm{Fix}(f)) < \varepsilon \}$ is open. $\rightarrow$ (iii) is cnvgr
- Increasing polynomials are dense in H_+^{AC}

- 

$$\psi_\phi^{a,b}(x) := \begin{cases} \frac{a}{\phi(a)}\,\phi(x) & x \in [0,a) \\ x & x \in (a,b) \\ 1 - \frac{1-b}{1-\phi(b)}\,(1-\phi(x)) & x \in (b,1] \end{cases}$$

$$\rho_{AC}(\phi, \psi_\phi^{(a,b)}) \leq |a - \phi(a)| + |b - \phi(b)| + (b-a) + (\phi(b) - \phi(a))$$